Ex Find the sum, (if possible) or show it diverges

blah

(1) $$\sum_{n=5}^{\infty} \frac{n^3+(-1)^n\sin(n)-n}{1000n^4+6}$$

$$\lim_{n\to\infty} \frac{1}{n}\cdot\frac{(1000n^4+6)}{(n^3+(-1)^n\sin(n)-n)}$$
$$=\lim_{n\to\infty}\frac{1000+\overbrace{6/n^4}^{\to 0}}{1+\underbrace{\frac{(-1)^n\sin(n)\cdot n-n^2}{n^4}}_{\to 0}}$$
$$=1000$$

since this # is nonzero, convergence of $\sum \frac{1}{n}$ and $\sum(\text{blah})$ is same. limit comparison test.

Since $\sum \frac{1}{n}$ diverges (harm. series) this sum also $\boxed{\text{diverges}}$.

(2) $$\sum_{n=5}^{\infty} \underbrace{\frac{n^3+(-1)^n\sin(n)-n}{1000\cdot 4^n+6}}_{c_n}$$

similar limiting behavior to $b_n = \frac{n^3}{1000\,4^n}$

$\lim \frac{b_n}{c_n} = 1 \to$ convergence or divergence depends on $\sum b_n$

$$\sum_{n=5}^{\infty}\underbrace{\frac{n^3}{1000\cdot 4^n}}_{b_n}$$

Root test $\lim_{n\to\infty}\sqrt[n]{|b_n|}$

$$=\lim_{n\to\infty}\frac{\sqrt[n]{n^3}}{\sqrt[n]{1000\,4^n}}$$

$$=\lim_{n\to\infty}\frac{\overbrace{n^{3/n}}^{\to 1}}{\underbrace{(1000)^{1/n}}_{\to 1}\cdot 4} = \frac{1}{4} < 1$$

$\Rightarrow \boxed{\text{Converges}}$

③ $$\sum_{k=2}^{\infty} \frac{2^k + 3^k + 4^k(-1)^k}{5^k}$$

geometric series

$$\sum_{k=2}^{\infty} \frac{2^k}{5^k} = \sum_{k=2}^{\infty}\left(\frac{2}{5}\right)^k = \frac{\frac{4}{25}}{1-2/5} = \frac{\frac{4}{25}}{3/5} = \boxed{\frac{4}{15}}$$

$$\sum_{k=2}^{\infty} \frac{3^k}{5^k} = \sum_{k=2}^{\infty}\left(\frac{3}{5}\right)^k = \frac{\frac{9}{25}}{1-3/5} = \frac{\frac{9}{25}}{2/5} = \boxed{\frac{9}{10}}$$

$$\sum_{k=2}^{\infty} \frac{4^k(-1)^k}{5^k} = \sum_{k=2}^{\infty}\left(\frac{-4}{5}\right)^k = \frac{\frac{16}{25}}{1-\left(-\frac{4}{5}\right)} = \frac{16}{25}\cdot\frac{5}{9} = \boxed{\frac{16}{45}}$$

$\boxed{\text{Ans: converges to } \frac{4}{15} + \frac{9}{10} + \frac{16}{45}}$

④ $$\sum_{m=1}^{\infty} \frac{(-1)^m (2.5)^m}{7 \cdot m!}$$

$$e^x = \sum_{m=0} \frac{x^m}{m!} = 1 + \sum_{m=1}^{\infty} \frac{x^m}{m!}$$

$$\sum_{m=1}^{\infty} \frac{x^m}{m!} = e^x - 1$$

$$\frac{1}{7}\sum_{m=1}^{\infty} \frac{x^m}{m!} = \frac{1}{7}(e^x - 1)$$

$$\sum_{m=1}^{\infty} \frac{(-2.5)^m}{7 \cdot m!} = \boxed{\frac{1}{7}\left(e^{-2.5} - 1\right)}$$

converges!

⑤ $\displaystyle\sum_{k=5}^{\infty} \frac{6\cdot 5^k + 2k^2}{1000\cdot 5^k + 2000\cdot 4^k} = \sum a_k$

$$= \sum_{k=5}^{\infty} \frac{6\cdot 1 + \frac{2k^2}{5^k}}{1000\cdot 1 + 2000\cdot\frac{4^k}{5^k}}$$

$\frac{2k^2}{5^k} \to 0$, $\quad \frac{4^k}{5^k} \to 0$

$$\lim_{k\to\infty} a_k = \frac{6}{1000} \neq 0.$$

$\therefore \sum a_k$ diverges.

Important fact:

If $\sum a_k$ converges, then $a_k \to 0$ as $k \to \infty$.

$\Leftrightarrow$ If $a_k \not\to 0$, then $\sum a_k$ diverges.

Note: converse is false

(eg $\sum \frac{1}{n}$ diverges even though $\frac{1}{n} \to 0$.)

Series Convergence:

- harmonic series diverges
- If $\sum a_k$ converges, then $\lim a_k = 0$
  [If $\lim a_k \neq 0$, then $\sum a_k$ diverges.]
- Comparison test
- limit comparison test
- integral test
- p-series test $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$
  diverges if $0 \leq p \leq 1$.
- alternating series test
- absolute convergence.
- ratio test
- root test

Taylor series to know

- general formula $f(a) + f'(a)(x-a) + \cdots$
- e^x
- $\frac{1}{1-x}$
- $\sin(x)$
- $\cos(x)$
- $\frac{a}{1-r}$
- $(1+x)^p$

Special limits:

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$

$$\lim_{t \to 0} \frac{e^t - 1}{t} = 1$$

$$\lim_{y \to \infty} \left(\frac{y+1}{y}\right)^y = e$$

Ex $$\int_0^{2\pi} \sin^4(x)\,dx = \int_0^{2\pi} \left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right)^2 dx$$

$$= \frac{1}{4}\int_0^{2\pi} \left(1 - \cos(2x)\right)^2 dx$$

$$= \frac{1}{4}\int_0^{2\pi} \left[1 - 2\cos(2x) + \underbrace{\cos^2(2x)}_{\frac{1}{2} + \frac{1}{2}\cos(4x)}\right] dx$$

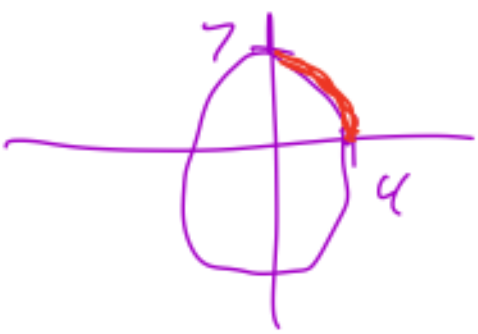
$$= \frac{1}{4}\int_0^{2\pi} \left[\frac{3}{2} - 2\cos(2x) + \frac{1}{2}\cos(4x)\right] dx$$

$$\downarrow \qquad \downarrow$$
$$0 \qquad 0$$

$$= \frac{3}{8}\cdot 2\pi = \boxed{\frac{3\pi}{4}}.$$

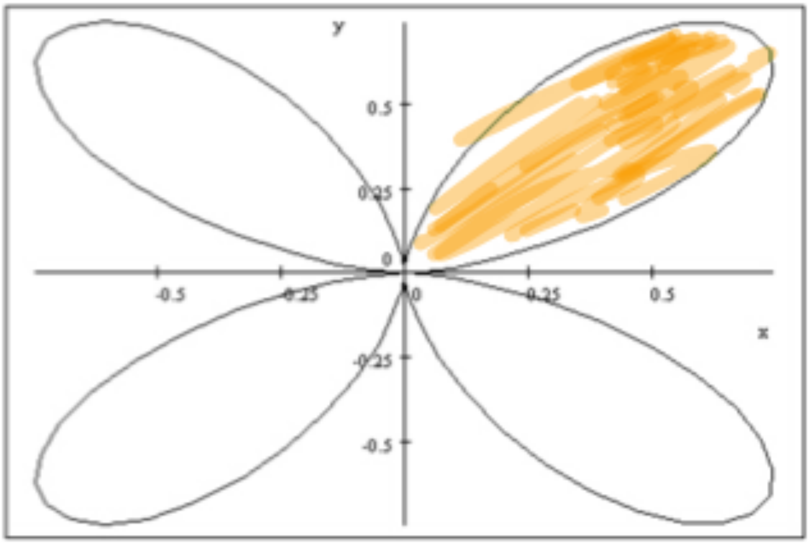
Ex Find the length of the ellipse $(4\cos(t), 7\sin(t))$ between $t=0$ & $t=\frac{\pi}{2}$

$(4,0)$ & $(0,7)$.



$$\text{Length} = \int_0^{\pi/2} \sqrt{(x')^2+(y')^2}\,dt$$

9.16 Find the area enclosed by one loop of the curve described in polar coordinates as $r = \sin^2(2\theta)$.



$r = 0 \Rightarrow \theta = 0$ or $\theta = \pi/2$

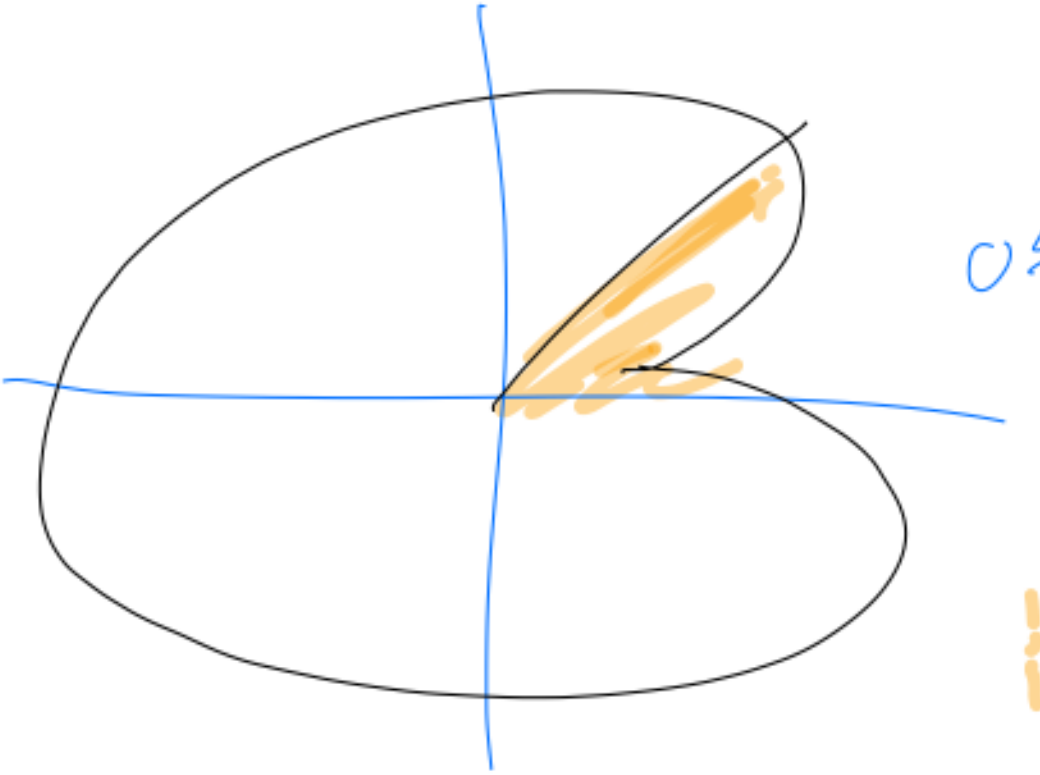
$\sin(2\theta) = 0$

$2\theta = 0$ or π

$$\text{Area} = \frac{1}{2}\int_0^{\pi/2} r^2\, d\theta = \ldots$$

(in this case, by symmetry

$$\text{Area} = \frac{1}{8}\int_0^{2\pi} r^2\, d\theta \text{)}$$



$0 \le \theta \le \frac{\pi}{4}$

$$\text{Area between} = \frac{1}{2}\int_0^{2\pi} r_1^2 \, d\theta - \frac{1}{2}\int_0^{2\pi} r_2^2 \, d\theta$$

$$\int_{a>0}^{\infty} \frac{1}{x^p}\,dx \quad \text{converges if } p>1, \quad \text{diverges if } p \le 1$$

$$\int_{0}^{a} \frac{1}{x^p}\,dx \quad \text{converges if } 0<p<1, \quad \text{diverges if } p \ge 1$$

To prove a sequence converges:

- This might work: Monotone Convergence Theorem (MCT).
  (x_n)

If the sequence is bounded $A \le x_n \le B$ $\forall n$.

& if the sequence is monotone

ie $x_{n+1} \ge x_n$ $\forall n$ (increasing)

or $x_{n+1} \le x_n$ $\forall n$ (decreasing.

Then MCT $\Rightarrow$ sequence converges.

$\boxed{\text{Ex}}$ Find the limit of this sequence, if it exists.

$$x_1 = 11, \quad x_{n+1} = \sqrt{12 + 7x_n} \quad \text{for } n \geq 1$$

We will show x_n satisfies
$$0 \leq x_n \leq 100.$$

Proof: $0 \leq x_1 = 11 \leq 100$. ✓ base case.

Now, suppose $0 \leq x_n \leq 100$ for some $n \geq 1$.

$$\Rightarrow 0 \leq 7x_n \leq 700$$

$$\Rightarrow 12 \leq 12 + 7x_n \leq 712$$

$$\Rightarrow 0 \leq \sqrt{12} \leq \underbrace{\sqrt{12 + 7x_n}}_{x_{n+1}} \leq \sqrt{712} < 100$$

$\therefore$ $0 \leq x_{n+1} \leq 100$ also.

$\therefore$ by induction $0 \leq x_n \leq 100$ for all n. ☺

$$(x_n) = \left(11, \sqrt{12+7\cdot 11} = \sqrt{89}, \sqrt{12+7\cdot\sqrt{89}}, \ldots\right)$$

Note $x_2 = \sqrt{89} \leq x_1 = 11$

Suppose $x_{n+1} \leq x_n$ for some $n \geq 1$.

$$\Rightarrow \quad 7x_{n+1} \leq 7x_n$$

$$\Rightarrow 12 + 7x_{n+1} \leq 12 + 7x_n$$

$$\Rightarrow \underbrace{\sqrt{12+7x_{n+1}}}_{x_{n+2}} \leq \underbrace{\sqrt{12+7x_n}}_{x_{n+1}}$$

$\therefore$ $x_{n+2} \leq x_{n+1}$ also.

by induction

$\therefore$ (x_n) is decreasing.

By MCT, (x_n) converges to a limit $L = \lim x_n = \lim x_{n+1}$

$$x_{n+1} = \sqrt{12 + 7\cdot x_n}$$

$$\Rightarrow \lim x_{n+1} = \lim \sqrt{12 + 7 \cdot x_n}$$

$$\Rightarrow L = \sqrt{12 + 7 \underbrace{\lim x_n}_{L}}$$

$$\Rightarrow L = \sqrt{12 + 7L}$$

$$\Rightarrow L^2 = 12 + 7L$$

$$\Rightarrow L^2 - 7L - 12 = 0$$

$$\Rightarrow L = \frac{7 \pm \sqrt{49 + 48}}{2}$$

$$\Rightarrow L = \frac{7 \pm \sqrt{97}}{2}$$

$L \geq 0$ since $x_n \geq 0 \ \forall n$

$$\Rightarrow \boxed{L = \frac{7 + \sqrt{97}}{2}}$$

Question:

Does $$\sum_{m=2}^{\infty} \frac{\sin(m)+3m}{\sqrt{m^5-m}} = \sum_{m=2}^{\infty} b_m$$ converge or diverge?

Note $$\frac{\sin(m)+3m}{\sqrt{m^5-m}} \approx \frac{3m}{\sqrt{m^5}} = \frac{3}{m^{1.5}}$$ for large m.

$$\lim_{m\to\infty} \frac{3}{m^{1.5}} \cdot \frac{\sqrt{m^5-m}}{(\sin(m)+3m)}$$

$$= \lim_{m\to\infty} \frac{3\,\cancel{m^{2.5}}\sqrt{1-\underbrace{\frac{1}{m^4}}_{\to 0}}}{\cancel{m^{2.5}}\left(\underbrace{\frac{\sin(m)}{m}}_{\to 0}+3\right)}$$

$$\sqrt{m^5-m} = \sqrt{m^5\left(1-\frac{1}{m^4}\right)} = \sqrt{m^5}\sqrt{1-\frac{1}{m^4}} = m^{2.5}\sqrt{1-\frac{1}{m^4}}$$

$= 1 \neq 0$. By the limit comparison test, $\sum_{m=2}^{\infty} b_m$ converges $\Leftrightarrow$ $\sum_{m=1}^{\infty} \frac{3}{m^{1.5}}$ converges

converges by p-series test
$p = 1.5 > 1.$

$\therefore \sum_{n=2}^{\infty} b_n$ converges.

$$\sum_{n=1}^{\infty} \frac{n^2+2}{n^2+\sqrt{n}} = \sum_{n=1}^{\infty} a_n$$

$$\lim_{n\to\infty} \frac{n^2+2}{n^2+n^{1/2}} = \lim_{n\to\infty} \frac{1+\frac{2}{n^2}}{1+\frac{1}{n^{3/2}}} = 1.$$

$\frac{2}{n^2} \to 0$, $\frac{1}{n^{3/2}} \to 0$

since $\lim a_n = 1 \neq 0$,

$\sum a_n$ diverges.